1. Ramsey model and qualitative analysis

Consider the Ramsey model in Problem 6 from Problem Set 1 with \( y = k^\alpha \) and \( U_0 = \int_0^{+\infty} \frac{e^{-\rho t} \left(c(t)\right)^\beta}{\rho} dt \), where \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \).

a) Describe how each of the following unexpected changes affects the locus of \( \dot{c} = 0 \) and \( \dot{k} = 0 \) in a \( k - c \) diagram, and how they affect the balanced-growth-path values of \( c \) and \( k \):
   i) A rise in \( \beta \);
   ii) A downward shift in the production function (a lower \( \alpha \));
   iii) A rise in the rate of depreciation;
   iv) A fall in the rate of time preference \( \rho \).

b) How do per-capita-consumption, capital intensity and the interest rate change during the adjustment processes if the initial capital intensity is at the initial steady state?

2. Ramsey model for decentralized economy

Consider a Ramsey-Cass-Koopmans economy that is already on its balanced growth path. Suppose that the government introduces a tax on investment \( \tau \) income at time \( t = 0 \), unexpectedly. Thus, the real interest rate that households face is now given by \( r(t) = (1 - \tau)f'(k(t)) \).

Assume that tax revenue is redistributed through lump sum transfers.

a) How does the tax affect the locus of \( \dot{c} = 0 \) and \( \dot{k} = 0 \)?

b) How does the economy respond to the adoption of the tax at \( t = 0 \), and what are the dynamics afterwards?

c) Suppose there are many economies like this one, distinguished by different tax rates.
   i) Show that the savings rate on the balanced growth path is decreasing in \( \tau \).
ii) Do citizens in high saving countries have an incentive to invest in low saving countries?

d) How, if at all, do the answers to part a) and b) change if the government does not rebate the tax revenue but instead uses it for government purchases?

Suppose that instead of announcing and implementing the tax at time $t = 0$, the government announces at $t = 0$ that it will begin to tax investment income at some later time $t_1$. Then

e) Draw the phase diagram showing the dynamics of $c$ and $k$ after time $t_1$.

f) Can $c$ change discontinuously at $t_1$? Why or why not?

g) Draw the phase diagram showing the dynamics of $c$ and $k$ before $t_1$.

h) What must $c$ do at time $t = 0$?

i) Summarize your results by sketching the paths of $c$ and $k$ as functions of time.

3. Ramsey model with technological progress

Suppose that, in a Ramsey-Cass-Koopmans economy, production is given by the function $Y_t = F(K_t, e^{\phi}L_t)$, where $\phi$ is the constant and exogenous rate of technical progress. Assume that the population grows at rate $n$ and that the utility function is of constant relative risk aversion form, with a coefficient of relative risk aversion equal to $\gamma$.

a) Derive and interpret the modified golden rule condition in this case.

b) Characterize the dynamics of consumption and capital accumulation.

c) Suppose that the economy is in steady state and that $\phi$ decreases permanently and unexpectedly. Describe the dynamic adjustment of the economy to this adverse supply shock.

4. Ramsey model with labor-leisure choice

Consider a Ramsey growth model where the representative consumer maximizes

$$U = \int_0^{+\infty} e^{-\rho t} u[c(t), l(t)] dt, \rho > 0$$

However, you may find it more straightforward if you rewrite it in the Hamiltonian approach.
with instantaneous preferences over consumption \( c(t) \) and hours of work \( l(t) \)

\[
    u[c(t), l(t)] = \ln c(t) - \theta \frac{l(t)^{1+\eta} - 1}{1 + \eta}, \theta > 0, \eta > 0.
\]

Workers have a time endowment of one unit of labour, so \( l(t) \in [0, 1] \) (Assume a constant population, normalised to 1.) Hours worked, \( l(t) \), are combined with physical capital, \( k(t) \), to produce output, \( y(t) \), through a Cobb-Douglas technology

\[
    y(t) = k(t)^\alpha l(t)^{1-\alpha}, \alpha \in (0, 1).\]

Capital accumulation in the economy follows

\[
    \dot{k}(t) = i(t) - \delta k(t)
\]

where \( i(t) \) is investment and capital depreciates at rate \( \delta \). The goods market clears at all times \( t \)

\[
    y(t) = c(t) + i(t).
\]

a) State the planner’s problem for this economy. Write down the Hamiltonian and derive the optimality conditions.

b) Express consumption and the capital stock in terms of units of labour, and call the new normalized variables \( \hat{c}(t) \) and \( \hat{k}(t) \). Are the steady-state levels of consumption \( \hat{c}(t) \) and capital \( \hat{k}(t) \) different from the standard Ramsey model without endogenous labour supply? Explain.

c) Derive the following expression for \( \dot{l} \) as a function of \( \frac{\dot{k}}{k} \) and \( \frac{\dot{c}}{c} \)

\[
    \dot{l} = \frac{1}{1+\eta} \frac{\dot{c}}{c} + \frac{\alpha}{1+\eta} \frac{\dot{k}}{k}.
\]

Provide some economic interpretation for this equation.

d) By using the dynamic equations of \( \frac{\dot{k}}{k} \) and \( \frac{\dot{c}}{c} \) show that

\[
    \frac{\dot{l}}{l} = \frac{\alpha}{\alpha + \eta} \left[ \frac{\dot{c}^*}{\dot{k}^*} - \frac{\dot{c}(t)}{\dot{k}(t)} \right].
\]

It is known that when \( \dot{k}(0) < \dot{k}^* \), \( \frac{\dot{c}(t)}{\dot{k}(t)} \) declines monotonically along the saddle path to the steady state. Explain whether the speed of convergence of \( \dot{k}(t) \) to the steady state is larger or smaller than for the standard Ramsey model.
5. Heterogeneity, capitalism, and inequality

This problem is an extension of the standard decentralized Ramsey-Cass-Koopmans model. Consider the market economy of Republic of Utopia, whose population is a continuum of measure \( L \). We keep most of the settings identical to the standard decentralized Ramsey-Cass-Koopmans model, except that the consumers in this economy are heterogenous:

- A share \( L_k \) of the population are Keynesian consumers, who are assumed simply to set their consumption equal to their current income;
- A share \( L_r = L - L_k \) of the population are Ramsey consumers, who are assumed to solve a traditional Ramsey-Cass-Koopmans dynamic optimization problem to determine their consumption.

To ease your computation, most of the steps are done for you in the following.

The aggregate production function is Cobb-Douglas in labor and capital, \( Y_t = K_t^\alpha L_t^{1-\alpha} \) and both capital and labor markets are perfect implying that interest rates and wages are given by

\[
\begin{align*}
r_t &= f'(k_t) - \delta, \\
w_t &= f(k_t) - k_t f''(k_t)
\end{align*}
\]

in which \( k_t \) is the capital intensity of the economy. Assume CRRA utility function \( u(c_t) = \frac{c_t^{1-\sigma}}{1-\sigma} \) for all consumers. Indicate per capita variables for the two types of consumers by a subscript, i.e. consumption per Ramsey consumer is \( c_{r,t} \), consumption per Keynesian consumer is \( c_{k,t} \), and so forth.

Therefore the optimal decision problem for a representative Ramsey consumer is

\[
\max_{\{c_{r,t},k_{r,t}\}_{t=0}} \int_{0}^{\infty} e^{-\rho t} u(c_{r,t}) dt,
\]

s.t. \( \dot{k}_{r,t} = \left[f'(k_t) - \delta\right] k_{r,t} + w_t - c_{r,t} \)

in which \( k_{r,t} \) is the capital intensity per Ramsey consumer and \( \delta > 0 \) is the constant depreciation rate. And as you have see from the definition, a representative Keynesian consumer blindly consumes everything she earns in each moment,

\[
c_{k,t} = w_t + f'(k_t)k_{k,t}, \quad (1)
\]

i.e. her labor income plus the products generated by her capital holdings. Here are the questions.
Part I: Equilibrium

a) Using Hamiltonian, derive the Euler equation for the Ramsey consumer.

b) Show that the steady-state capital stock in this model is the same as in an economy entirely populated by Ramsey consumers.

c) Discuss verbally the plausibility of this result if the proportion of the population that is Ramsey is very small (but different from zero).

Part II: Inequality

Suppose that \( L_r = L_k = \frac{1}{2} \) and the economy is already in its steady state. Economists care about inequality of this economy, which may be captured by the following three measures. Compute these three measures with \( \alpha = \frac{1}{3} \) and \( L = 1 \).

- The fraction of total labor income going to the poor (the Keynesian consumers);
- The poor’s fraction of aggregate capital holdings;
- The poor’s fraction of aggregate income (Remember that the income of anyone in this economy includes her wage and the products generated by her capital holdings, as the right hand side of (1) shows).

Part III: Policy change

Some economists of Utopia argue that the poor are Keynesian, blindly consuming every cent of their income, because they do not have access to financial markets so that they are physically unable to save. Suppose the economy of Utopia is already in its steady state and the policy maker starts a policy change such that Keynesian consumers are suddenly granted access to financial markets and able to make savings from now on. Discuss verbally the dynamics of inequality.

6. Overlapping generations: Dynamic inefficiency

Consider the following economy of overlapping generations of vegetarians. Assume there is an equal number of young and old individuals and population \( N \) is constant. Each individual is endowed with one vegetable at birth. While young, the individual decides how much of the vegetable to eat, and how much to plant (there are no refrigerators so it can not be stored). If a fraction \( s \) of the vegetable is planted, then \( As^\alpha \) vegetables will grow for second-period consumption, in which \( A > 0 \) and \( 0 < \alpha < 1 \). The portion planted cannot be eaten (this is equivalent to 100% depreciation). Utility is of the form:

\[
 u(c_{1t}, c_{2t+1}) = \ln c_{1t} + \ln c_{2t+1}.
\]
a) Write down the individual’s decision problem. How much does the individual consume and save in the first period?

b) Can trade between generations take place in this economy? Describe the equilibrium in this economy.

c) Suppose that in each period the government confiscates a fraction $f$ of each young person’s vegetable, and distributes the proceeds equally among the old individuals. Find expressions for first and second period consumption, $c_{1t}$ and $c_{2t+1}$, in this case.

d) Consider a small increase in $f$ starting from $f = 0$. Derive an expression for welfare as a result of this policy.

e) Under what conditions is the policy in part d) welfare improving? Provide a brief intuitive explanation.


7. Overlapping generations with money

(Samuelson, 1958) Suppose, as in the Diamond (1965) model, that $N_t$ 2-period-lived individuals are born in period $t$ and that generations are growing with rate $n$. The utility function of a representative individual is $U_t = \ln c_{1t} + \ln c_{2t+1}$. Each individual is born with an endowment of $A$ units of the economy’s single good. The good can either be consumed or stored. Each unit stored yields $x > 0$ units next period.

In period 0, there are $N_0$ young individuals and $\frac{1}{1+n}N_0$ old individuals endowed with some amount $Z$ of the good. Their utility is simply $c_{20}$.

a) Describe the decentralized equilibrium of this economy. (Hint: Will members of any generation trade with members of another generation?)

b) Consider paths where the fraction of agents endowment that is stored, $s_t$, is constant over time. What is per capita consumption (weighted average from young and old) on such a path as a function of $s$?

c) If $x < 1 + n$, which value of $s \in [0, 1]$ is maximizing per capita consumption?

d) Is the decentralized equilibrium Pareto-efficient? If not, how could a social planner raise welfare?

Suppose now that old individuals in period 0 are also endowed with $M$ units of a storable, divisible commodity, which we call money. Money is not a source of utility. Assume $x < 1 + n$.

e) Suppose the price of the good in units of money in periods $t$ and $t+1$ is given by $P_t$ and $P_{t+1}$, respectively. Derive the demand functions of an individual born in $t$.

f) Describe the set of equilibria.
g) Explain why there is an equilibrium with $P_t \to +\infty$. Explain why this must be the case if the economy ends at some date $T$ that is common knowledge among all generations.

References


