Macroeconomics (Research) (WS10/11)
Problem Set 3
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1. Capital adjustment costs

Remember that in standard growth models such as Solow-Swan and Ramsey-Cass-Koopmans, the output of the economy is simply split between consumption and saving, and the households’ savings are then transformed into the firms’ investments, which become the increments in the capital stock after depreciation. However, reality seems to be much more complicated and empirical data doesn’t match what the theories predict at all — For example, in reality the convergence speed is much slower and the volatility in investments is three times as high as that of output. Therefore, the motivations for saving and investment need to be examined more carefully.

In the following we develop a partial equilibrium model to explain the firms’ motives of investment, arising from the internal adjustment costs. Suppose our economy is based on the neoclassical production function $Y = F(K, L)$, i.e. the economy as a whole can be regarded as a single firm. There is no technological progress, and the depreciation rate is constant at $\delta$.

Assume that with accumulation of capital, convex adjustment costs have to be incurred (as an increasing function of investment relative to the capital stock) such that the cost of investment becomes $I(t) \left[ 1 + \Phi \left( \frac{I(t)}{K(t)} \right) \right]$ with $\phi(0) = 0$, $\phi'(\cdot) > 0$, $\phi''(\cdot) > 0$. Assume that instead of leasing capital from the households, the firms own the capital, and there is a constant interest rate $r$.

Therefore, the firms’ problem is to maximize the profit in an infinite time horizon, such that

$$\max_{\{K(t), I(t), L(t)\}_{t=0}^{\infty}} \int_{t=0}^{\infty} e^{-r(t)} \left\{ F(K(t), L(t)) - w(t)L(t) - I(t) \left[ 1 + \Phi \left( \frac{I(t)}{K(t)} \right) \right] \right\} dt,$$

(1)

and the law of motion in capital flow is given by the equality

$$\dot{K}(t) = I(t) - \delta K(t).$$

(2)

a) Formulate the current value Hamiltonian from (1) & (2), and get the first order conditions.

b) Characterize the path for optimal capital accumulation in the phase diagram of $q - K$
space.

c) Explain why the shadow price of capital \( q \) exceeds 1 and may differ from the one in a steady state.

\[ \text{Barro and Sala-i-Martin (2004), Chapter 3.2.} \]

### 2. Sustainability of debt in a small open economy

Consider a small open economy with an infinitely lived representative consumer with CES utility, the instantaneous elasticity of substitution being \( \sigma \) and discount rate \( \rho \). The economy’s GDP growth rate is \( \frac{\dot{Y}}{Y} = y \), the growth rate of population is a constant \( n \), and the interest rate is a constant \( r \). Initial asset positions are zero.

a) Derive the present value budget constraint. Assume that the representative consumer is a Ramsey consumer, i.e. her optimal consumption path follows that in Ramsey-Cass-Koopmans model. Under which conditions is this optimal consumption path well defined?

b) Compute the interest rate in autarky.

c) Suppose that the country is initially running a current account deficit. How are current account and debt / GDP-ratio evolving over time?

\[ \text{Obstfeld and Rogoff (1996), Chapter 2A.1 Deriving the Steady-State Debt-Output Ratio.} \]

### 3. Sustainability of government debt

Consider an economy with constant GDP growth rate \( y \) and interest rate \( r > y \). The government finances its expenditure \( G \) through tax income \( T \) and debt \( B \).

a) Suppose that the government’s primary deficit rate \( d_p = \frac{G - T}{Y} \) is given exogenously and constant over time. Show how the debt / GDP ratio (suppose that the initial value of this ratio is \( b_0 \)) evolves over time. Which restrictions are needed to ensure long-run stability of the debt / GDP ratio? How does your answer change, if \( r < y \)?

b) Suppose that the government’s total deficit rate \( d_t = \frac{G - T + rB}{Y} \) is given exogenously and constant over time. Show how the debt / GDP ratio evolves over time. Which restrictions are needed to ensure long-run stability of the debt / GDP ratio? How does your answer depend on the interest rate?

c) What are the long-run implications of a constant total deficit rate for the primary deficit?

d) Suppose now that \( r \) is the nominal interest rate and \( y \) is the growth rate of nominal output.
How would a rise in the rate of inflation affect the long-run primary deficit if the total deficit rate is held constant?

e) Discuss the relation of your results with the Maastricht criteria.

Obstfeld and Rogoff (1996), Chapter 2A.1 Deriving the Steady-State Debt-Output Ratio.

4. Stochastic optimization: Asset pricing

Consider a household with expected utility function

\[ E[U] = U(c_t) + \frac{1}{1+\rho} \sum_s p_s U(c_{2,s}), \]

where \( p_s \) is the probability of state \( s \). Income is \( y_1 \) in the first period and \( y_{2,s} \) in state \( s \) of period 2. There is one asset traded in period 1 that pays an interest rate of \( r \) in each state of period 2.

a) Write down budget constraints and derive the first order condition. Show that for optimal savings the asset’s return equals the expected marginal rate of substitution between present and future consumption.

Assume a CRRA utility function \( U(c) = \frac{c^\alpha}{\alpha} \) and assume that the stochastic income in period 2 is such that with optimal savings the MRS has log-normal distribution, i.e. \( \ln(MRS) \sim N(\mu, \sigma^2) \).

b) Show that the difference between interest rate and time preference rate rises with increasing variance.

Consider now an asset with stochastic return \( 1 + r_s \).

c) Write down budget constraints and derive the first order condition. Show how the asset price depends on the covariance between \( r_s \) and \( y_{2,s} \).

Cochrane (2005), Chapter 1.

5. Stochastic optimization: Permanent income hypothesis

Consider a consumer maximizing expected utility in discrete time under uncertainty subject to a budget constraint. The interest rate \( r \) is constant, as is the rate of time preference \( \rho \). The consumer has an initial stock of assets \( A \) and earns income an \( Y_t \) that is uncertain in the future.
In each period, the consumer solves the following problem:

\[
\max_{\{C_t\}_{t=0}^{\infty}} E_t \left[ \sum_{t=0}^{\infty} \frac{1}{(1 + \rho)^t} U(C_t) + \lambda \left( A - \sum_{t=0}^{\infty} \frac{1}{(1 + r)^t} (C_t - Y_t) \right) \right],
\]

in which \( \lambda \) is the Lagrange multiplier.

a) Using the first order conditions, show that marginal utility is a random process of the form

\[X_{t+1} = kX_t + \epsilon_{t+1},\]

where \( X \) denotes marginal utility, \( k \) is a constant, and \( \epsilon \) is a random term with mean zero.

b) Show that for quadratic utility, \( U(C) = -\frac{(b-C)^2}{2} \), consumption is a stochastic process of the form

\[C_{t+1} = kC_t + \delta + \epsilon_{t+1},\]

where \( k \) and \( \delta \) are constants, and \( \epsilon \) is a random term with mean zero.

Cochrane (2005), Chapter 1.

6. Asset pricing: The Lucas tree

(Lucas, 1978) Suppose that the only assets in the economy are some infinitely living trees. Output equals the fruits of the trees (suppose the productivities of the trees are perfectly correlated, i.e. all the trees produce exactly the same amount of fruits in a given period), which is exogenously given positive random variable and cannot be stored — therefore \( c_t = y_t \) for each \( t \) in which \( y_t \) is the exogenously determined per capita output (to make it simple, one can assume that the number of trees is equal to the population, i.e. \( y_t \) is also the productivity of the trees in period \( t \)) and \( c_t \) is the per capita consumption. Assume that in the beginning each one in this economy owns the same number of trees. Since all the agents are assumed to be the same, in equilibrium the behavior of the price of the trees should be such that in each period the representative agent is not willing to either increase or decrease her holdings of the trees.

Let \( P_t \) denote the price of a tree in period \( t \), and assume that if the tree is sold the sale occurs after the existing owner receives that period’s output. Finally, assume that the representative agent maximizes

\[E_0 \left[ \sum_{t=0}^{\infty} \frac{1}{(1 + \rho)^t} \frac{c_t^{1-\sigma}}{1-\sigma} \right].\]

a) Suppose that the representative agent reduces her consumption is period \( t \) by an infinitesimal amount, uses the resulting saving to increase her holdings of trees and then sells these additional trees in period \( t + 1 \). Find the condition that \( c_t \) and expectations involving \( y_{t+1}, P_{t+1} \) and \( c_{t+1} \) must satisfy for this change not to affect expected utility. Solve this condition for \( P_t \)
in terms of \(y_t\) and expectations involving \(y_{t+1}, P_{t+1}\) and \(c_{t+1}\) (Hint: The representative agent’s resource constraint can be written as \(c_t + P_t e_t + q_t b_{t+1} = (y_t + P_t) e_t\), in which \(e_t\) denotes how many trees she owns in period \(t\).)

b) Suppose that \(\sigma \to 1\) and \(\lim_{s \to +\infty} E_t \left[ \frac{1}{(1 + \rho)^s} \frac{P_{t+s}}{y_{t+s}} \right] = 0\). Iterate the result in a) forward to solve for \(P_t\).

c) Give some intuition why in b) an increase in expectations of future dividends does not affect the price of the asset.

d) Does consumption follow a random walk in this model?


7. The equity premium puzzle

(Mehra and Prescott, 1985) Continue with the settings in Problem 6 a). Now except the ownership of the tree, we introduce another asset — a riskless asset \(b_t\) with price \(q_t\). We call such riskless asset bond, and the risky assets (the ownership of the trees) equity or stock.

a) Define the representative agent’s optimization problem, and derive the first order conditions. Note that in addition to the agent’s flow budget constraint that you specified in Problem 6, in each period \(t\) the agent now has to decide how much \(b_{t+1}\) she has to invest for the next period at current price \(q_t\) (Hint: The representative agent’s resource constraint can be written as \(c_t + P_t e_t + q_t b_{t+1} = (y_t + P_t) e_t + b_t\).)

b) Express \(P_t\) and \(q_t\) in terms of \(y_t\) and expectations involving \(y_{t+1}, P_{t+1}\) and \(c_{t+1}\). Show how \(P_t\) depends on the covariance between \(P_{t+1}\) and \(c_{t+1}\), and define \(P_t\) as the sum of the riskless return and the risk premium.

c) Define the implicit return of the riskless asset, the bond, as

\[ R_b = \frac{1}{q_t} \]

and the implicit return of the risky assets, the stock, as

\[ R_s = \frac{P_{t+1} + y_{t+1}}{P_t} \]

Rewrite the expressions in b) with \(R_b\) and \(R_s\).

Now assume that the consumption growth rate is

\[ \frac{c_{t+1}}{c_t} = \frac{y_{t+1}}{y_t} = \gamma \exp \left( \epsilon_y - \frac{\sigma^2}{2} \right) \]
in which $\gamma$ is a positive constant and $\epsilon_{yt}$ is a normally distributed i.i.d. shock, $\epsilon_{yt} \sim N(0, \sigma_{y}^{2})$.

And assume that $R_s$ fluctuates around $\bar{R}_s$ as

$$R_s = \bar{R}_s \exp\left(\epsilon_{st} - \frac{\sigma_{s}^{2}}{2}\right),$$

in which $\epsilon_{st}$ is a normally distributed i.i.d. shock, $\epsilon_{st} \sim N(0, \sigma_{s}^{2})$.

Find the equity premium $\bar{R}_s - R_b$ in terms of $\sigma$, $\epsilon_{yt}$ and $\epsilon_{st}$.

d) Estimated from the US financial market (1890 – 2003), $R_b = 1.01$, $\bar{R}_s = 1.07$, and $\text{cov}(\epsilon_{st}, \epsilon_{yt}) = 0.002$. Compute $\sigma$ using the result of e). What does $\sigma$ mean in economics? Why do people call this result a puzzle?

According to Campbell (1999), Section 3.

References


